PROBABILISTIC CONSTRUCTION AND PROPERTIES OF GAMMA PROCESSES AND EXTENSIONS

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The point of this talk is to make an overview of probabilistic properties of gamma processes, and to present a few univariate and multivariate extensions.

In the oldest reliability literature, lifetimes of industrial systems or components were usually modeled through random variables, e.g. see [3] for a pioneer work on the subject. Based on the development of on-line monitoring which allows for the effective measurement of a system deterioration, numerous papers nowadays model the degradation in itself, which is often considered to be accumulating over time. In such a context, one can find shock models, where the deterioration is suddenly increased at each shock, in contrast to wear models, where the degradation level appears as mostly continuous. Deterioration induced by isolated shocks is typically modeled through compound Poisson or shot noise processes [19] whereas the most classical wear models are Wiener process (with trend) [10,19,25] and gamma process, the use of which seems to go back to the middle of the 80's [1,8]. Inverse gaussian process also appeared as a possible model more recently [24,26], as well as inverse gamma process, see [15,16]. This talk is devoted to the seemingly most common model, that is gamma process (and extensions). We refer to [23] for a comprehensive presentation and overview of applications of the gamma process to reliability theory, and to [17] for a deep account on its probabilistic construction and jump behavior (together with statistical inference procedures).

Given an increasing and continuous function $A : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $A(0) = 0$ and $b > 0$, let us first recall that a càdlàg process $\mathbf{X} = (X_t)_{t\geq 0}$ on $(\Omega, \mathcal{A}, \mathbb{P})$ is said to be a (non-homogeneous) gamma process with shape function $A(\cdot)$ and scale parameter b (written $\mathbf{X} \sim \mathcal{G}(A(\cdot), b)$) if

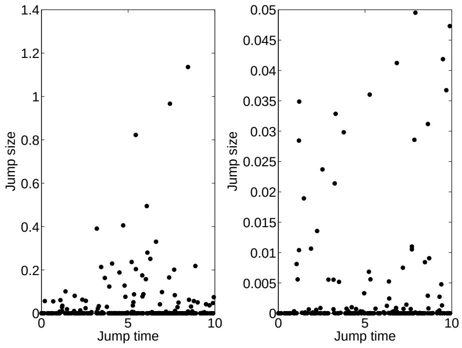

Figure 1. An observation of the jump times (U_n) and jump sizes (V_n) of a gamma process $\mathcal{G}(t,1)$, restricted to the jumps with size greater than 10^{-10} (left) and zoomed observation of the left plot, restricted to the jumps with size in $[10^{-10}, 0.05]$ (right)

- $X_0 = 0$ almost surely,
- $\mathbf{X}$ has independent increments,
- for all $0 \leq s < t$, the random variable $X_t - X_s$ is gamma distributed $\mathcal{G}(A(t) - A(s), b)$ with the following probability density function with respect to Lebesgue measure:

$$f(x) = \frac{b^{A(t)-A(s)}}{\Gamma(A(t) - A(s))} x^{A(t)-A(s)-1} e^{-bx} \; \mathbf{1}_{\mathbb{R}_+}(x), \text{ for all } x \in \mathbb{R}.$$

In the specific case where $A(t) = at$ with $a > 0$, the process $\mathbf{X}$ is said to be a homogeneous gamma process. It that case, it is a Lévy process [5]. In the most general case, it is an additive (or non homogeneous Lévy) process [22].

As a first step, probabilistic constructions of a gamma process will be reviewed through series representations of the shape

$$X_t = \sum_{n \geq 1} V_n \; \mathbf{1}_{[0,t]}(U_n),$$

based on the works of Bondesson [6] and Rosiński [21]. These constructions allow to give a good account of the jump structure of a gamma process $(X_t)_{t \geq 0}$, which can be seen to have an infinite activity [12], namely to jump infinitely many on any finite time interval. This is illustrated in Figure 1, where we can observe that there is an accumulation of jumps of small size. Based on this, the gamma process is well adapted to model deterioration that is accumulating over time as the result of many tiny increments [23].

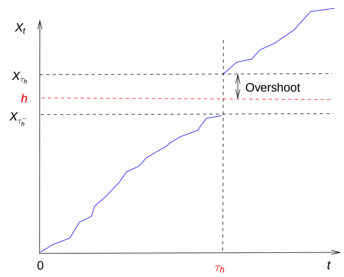

Figure 2. Overshoot of the gamma process

From an applicative point of view, a system with non-decreasing deterioration $(X_t)_{t\geq 0}$ is typically considered as failed (or too degraded) as soon the deterioration level is beyond a given failure threshold (say h). The time-to-failure of the system hence is the hitting time τ_h of the Borel set $[h, +\infty)$ by the process $(X_t)_{t\geq 0}$, with

$$\tau_h = \inf\left(t \geq 0 : X_t \geq h\right).$$

Based on the monotony of a gamma process, it is immediate to get the survival function of τ_h :

$$\bar{F}_{\tau_h}\left(t\right) = \mathbb{P}\left(\tau_h > t\right) = \mathbb{P}\left(X_t \leq h\right) = F_{X_t}\left(h\right),$$

where F_{X_t} is the cumulative distribution function of X_t.

However, contrary to what happens with a continuous process such as a Wiener process, the level h is not exactly reached at time τ_h : it is crossed by a jump, so that the level X_{τ_h} at time τ_h is a.s. larger than h, leading to an a.s. positive overshoot $(X_{\tau_h} - h)$, see Figure 2. In an applicative context, the failure may be all the more severe as the overshoot is higher, which may entail safety problems. Whence the interest of studying the after-jump level X_{τ_h} together with the jump time τ_h, as well as the size of the jump $X_{\tau_h} - X_{\tau_h^-}$. Several results will be reviewed on these hitting times, together with the before/after jump levels, based on results from [5]. Aging properties of τ_h will also be mentionned, which justify the use of preventive maintenance actions for a gamma deteriorating process.

A few properties of a gamma process will next be pointed out, which can be restrictive in an applicative context. Several extensions from the literature will be reviewed, which allow to overcome such limitations. As an example, the variance-to-mean ratio of a (standard) gamma process is known to be constant over time. This has lead to the development of Extended gamma process, as defined by [9], where the scale parameter may vary over time. See [14] for a practical use in reliability in a discrete time context, and [2] for approximate simulation methods and probabilistic computations. This process is also called Weighted gamma process by [11] (in a Bayesian context).

Based on the independence of its increments, another possible restriction of the gamma process is that a deterioration increment $X_t - X_s$ (with $0 < s < t$) does not depend on the previous history of the process at time s, whatever severe the deterioration could have been before s. A second extension called Transformed gamma process was recently proposed by [13], which overcomes this restriction.

A third extension called Pertubed gamma process is studied by [7], which allows to model non strictly monotonous deterioration (but with monotonous trend). This can, e.g., allow to deal with measurement errors.

Finally, the development of online monitoring often allows to measure several deterioration indicators at the same time, see, e.g., [20] for a practical example for railway studies. Apart from the previously mentioned univariate extensions, there hence is a crucial need for multivariate deterioration model. A possibility is to consider multivariate (non homogeneous) Lévy processes with gamma marginal processes. A few possible constructions from the literature will be reviewed, such as, e.g., superposition or subordination of independent gamma processes [4], or construction through Lévy copulas [18], which allows for more flexibility in the dependence of the marginal gamma processes.

Bibliography

[1] M. Abdel-Hameed. A gamma wear process. *IEEE Trans. Rel.*, 24(2):152–153, 1975.

[2] Z. Al Masry, S. Mercier, and G. Verdier. Approximate simulation techniques and distribution of an extended gamma process. *Methodol. Comput. Appl. Probab.*, 19(1):213–235, 2017.

[3] R. E. Barlow and F. Proschan. *Mathematical theory of reliability*, volume 17 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1965. With contributions by Larry C. Hunter, 1996.

[4] O. Barndorff-Nielsen, J. Pedersen, and K. Sato. Multivariate subordination, self-decomposability and stability. *Adv. in Appl. Probab.*, 33(1):160–187, 2001.

[5] J. Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.

[6] L. Bondesson. On simulation from infinitely divisible distributions. *Adv. in Appl. Probab.*, 14(4):855–869, 1982.

[7] L. Bordes, C. Paroissin, and A. Salami. Parametric inference in a perturbed gamma degradation process. *Commun. Statist – Theor. Meth.*, 45(9):2730–2747, 2016.

[8] E. Çinlar, Z. P. Bažant, and E. Osman. Stochastic process for extrapolating concrete creep. *J. of the Engrg. Mech. Div., ASCE*, 103(EM6):1069–1088, 1977.

[9] E. Çinlar. On a generalization of gamma processes. *J. Appl. Probab.*, 17(2):467–480, 1980.

[10] K. A. Doksum and A. Hyland. Models for variable-stress accelerated life testing experiments based on Wiener processes and the inverse gaussian distribution. *Technometrics*, 34(1):74–82, 1992.

[11] R. Dykstra and P. Laud. A bayesian nonparametric approach to reliability. *Ann. Statist.*, 9(2):356–367, 1981.

[12] E. Eberlein. Jump-type Lévy processes. In T. Mikosch, J.-P. Kreiß, R. A. Davis, and T. G. Andersen, editors, *Handbook of Financial Time Series*, pages 439–455. Springer Berlin Heidelberg, 2009.

[13] M. Giorgio, M. Guida, and G. Pulcini. A new class of markovian processes for deteriorating units with state dependent increments and covariates. *IEEE Trans. Rel.*, 64(2):562–578, 2015. cited By 0.

[14] M. Guida, F. Postiglione, and G. Pulcini. A time-discrete extended gamma process for time-dependent degradation phenomena. *Reliab. Eng. Syst. Saf.*, 105(0):73 – 79, 2012.

[15] M. Guida and G. Pulcini. The inverse gamma process: A family of continuous stochastic models for describing state-dependent deterioration phenomena. *Reliab. Eng. Syst. Saf.*, 120(0):72 – 79, 2013.

[16] B. P. Harlamov. On statistics of inverse gamma process as a model of wear. In *Probability, statistics and modelling in public health*, pages 187–201. Springer, New York, 2006.

[17] W. Kahle, S. Mercier, and C. Paroissin. *Degradation Processes in Reliability.* John Wiley & Sons, Inc., Hoboken, NJ, USA, 2016.

[18] J. Kallsen and P. Tankov. Characterization of dependence of multidimensional Lévy processes using Lévy copulas. *J. Multivariate Anal.*, 97(7):1551–1572, 2006.

[19] A. J. Lemoine and M. L. Wenocur. On failure modeling. *Naval Research Logistics*, 32:497508, 1985.

[20] S. Mercier, C. Meier-Hirmer, and M. Roussignol. Bivariate gamma wear processes for track geometry modelling, with application to intervention scheduling. *Struct. Infrastruct. E.*, 8(4):357–366, 2012.

[21] J. Rosiński. Series representations of Lévy processes from the perspective of point processes. In *Lévy processes*, pages 401–415. Springer, 2001.

[22] K. Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press,

Cambridge, 1999.
[23] J. Van Noortwijk. A survey of the application of Gamma processes in maintenance. *Reliab. Eng. Syst. Saf.*, 94(1):2–21, 2009.
[24] X. Wang and D. Xu. An inverse gaussian process model for degradation data. *Technometrics*, 52(2):188–197, 2010.
[25] G. A. Whitmore. Estimating degradation by a Wiener diffusion process subject to measurement error. *Lifetime Data Anal*, 1(3):307–319, 1995.
[26] Z.-S. Ye and N. Chen. The inverse gaussian process as a degradation model. *Technometrics*, 56(3):302–311, 2014.